Lecture 11: More on basis and dimension

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Recall:
size of any LI set in V $\leq$ size of any spanning set in V

9.4

Definition
Let V be a vector space and $\{v_1, \ldots, v_m\}$ be a set of vectors in V. $\{v_1, \ldots, v_m\}$ is called a **basis** of V if it is linearly independent and it spans V.

- a basis is:
  - a linearly independent spanning set of V
  - the biggest possible LI set in V
  - the smallest possible spanning set in V

Examples:

a) $\{(1,0),(0,1)\}$ and $\{(1,2),(3,4)\}$ are both bases of $\mathbb{R}^2$
b) $\{x^2, x, 1\}$ is a basis of $\mathbb{P}_2$
c) $\{(1,0)\}$ and $\{(1,0),(0,1),(1,1)\}$ are not bases of $\mathbb{R}^2$

Theorem
If $\{v_1, \ldots, v_m\}$ and $\{w_1, \ldots, w_n\}$ are two bases for a vector space V, then $m = n$.

PROOF
$\{v_1, \ldots, v_m\}$ is linearly independent, $\{w_1, \ldots, w_n\}$ meaning $m \leq n$, and $n \leq m$, so $m = n$.

Definition
If V has a finite basis $\{v_1, \ldots, v_m\}$, then the dimension of V is n, dim(V)=n. If V doesn't have a finite basis, it is **infinite dimensional**.

9.5 Examples

a) $\dim(\mathbb{R}^2)=2$
because $\{(1,0),(0,1)\}$ is a basis of $\mathbb{R}^2$
b) $\dim(\mathbb{R}^3)=3$, $\dim(\mathbb{R}^n)=n$
c) $\dim(M_{22}(\mathbb{R})) = 4$, $\dim(M_{mn}(\mathbb{R})) = mn$
d) $\dim(\mathbb{P}_2) = 3$, $\dim(\mathbb{P}_n) = n+1$
e) $\mathbb{P}$ and $F(\mathbb{R})$ are infinite dimensional
f) L=$\left\{\begin{bmatrix} a & b \\ c & -a \end{bmatrix} \mid a, b, c \in \mathbb{R}\right\}$ has basis $\left\{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right\}$, so dim(L)=3.
g) L=$\{(x+y, x+y, z) \mid x, y, z \in \mathbb{R}\}$
$= \text{span}\{(1,1,0), (1,1,0), (0,0,1)\}$
$= \text{span}\{(1,1,0), (0,0,1)\}$
So, dim(L)=2.
h) L= $\{(x, y, -x) \mid x, y \in \mathbb{R}\}$
$= \text{span}\{(1,0,-1), (0,1,0)\}$
So, dim(L)=2.

Problem:
Find a basis for $W = span\{1, \sin(x), \cos(x)\}$, a subspace of $F(\boldsymbol{R})$
Solution:
Right away, we know that $\{1, \sin(x), \cos(x)\}$ is a spanning set for W, and that's it's linearly independent (which we verified earlier). Therefore it's a basis for W and dim(W)=3.

Problem:
Find a basis for $U = \{(x, y, z) \mid x + z = 0\}$

Solution:
First, find a spanning set:

$$U = \{(x, y, -x\} \mid x, y \in \mathbb{R}\} = span\{(1,0,-1), (0,1,0)\}$$

$\{(1,0,-1), (0,1,0)\}$ spans U and is linearly independent, and therefore is a basis and dim(U)=2.

Problem:
Give 2 distinct bases for the following subspace, and hence give the dimension of the subspace.

$W = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{22}(\mathbb{R}) \;\middle|\; a + d = 0\right\}$

Solution:
Note that d=-a, so $W = \left\{\begin{bmatrix} a & b \\ c & -a \end{bmatrix} \in M_{22}(\mathbb{R}) \;\middle|\; a + d = 0\right\} = span\left\{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right\}$.

$a\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$.

This implies that $a = b = c = 0$, and so it is linearly independent, and therefore a basis of W.

$\left\{\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}\right\}$ is therefore another basis of W.

Therefore, dim(W)=3.